Grading guide, Pricing Financial Assets, August 2013

1. Consider a stock with current price S_0 that pays a constant dividend yield of q. Let C be the price of an American call, c the price of a European call, P the price of an American put, and p the price of a European put, all written on this stock and with the same exercise price K and maturity T. Assume a constant interest rate of r.
   (a) Show that $C \geq c$. What are sufficient conditions for $C = c$?
   (b) Under what general conditions may $P = p$ (No formal derivation is expected)?
   (c) Derive the call-put-parity for the European option prices
   (d) Show that
   $$S_0 \mathrm{e}^{-qT} - K \leq C - P \leq S_0 - K\mathrm{e}^{-rT}$$

Solution:

(a) The weak inequality follows directly (the American could just be held to maturity), whereas sufficient conditions for the American option not optimally exercised before maturity are no dividends and a positive interest rate. This can be shown by a simple arbitrage argument.

(b) If the American option is out-of-the-money it is not optimally exercised early. If it is in-the-money it may optimally be exercised early if the interest rate is high (and the dividend yield low). In other cases the European and American put will have the same price (a precise characterization of the optimal exercise boundary is not in the syllabus).

(c) This is a simpel arbitrage argument, where payments at the maturity date can be compared. See e.g. Hull p.351.

(d) This is slightly more complicated because of the American features. For the first inequality consider two portfolios (A) and (B):

- (A): One European call and K invested at the risk free rate
- (B): One American put and e^{-qT} shares with dividends reinvested in the stock

Start by considering the payoff at maturity, assuming the put is not exercised early. If $S_T > K$ the payoff is

- (A): $(\mathrm{e}^{rT} - 1)K + S_T$
- (B): S_T

and if $S_T \leq K$

- (A): $K\mathrm{e}^{rT}$
- (B): K

To complete the argument note that at any early exercise when the put is in-the-money the payoff of (B) is K which is lower than the present value of the payoff of (A) in either case. For arbitrage free prices we thus must have

$$C + K \geq c + K \geq P + S_0\mathrm{e}^{-qT}$$

The second part of the inequalities follows in a similar fashion.

2. (a) Describe the payment structure of a Credit Default Swap (CDS).

(b) Consider a tranched CDS or synthetic CDO. Explain the payment structure and define the terms *attachment point* and *detachment point*.

(c) Consider a tranched CDS or synthetic CDO on a large portfolio on underlying issuers (names). For a given average level of credit risk, e.g. expressed by the credit spread on the portfolio, explain how different levels of the assessed correlation of defaults of the issuers will influence the relative pricing of the tranches.

Solution:

(a)

Definition 0.1 (Credit Default Swap)**.** *A* ***Credit Default Swap*** *(CDS) is a contract between a protection seller and a protection buyer based on a specified Nominal Principal. The protection buyer pays a running premium (the CDS spread) until maturity of the contract or a Credit Event on the Reference Entity of the CDS. At a Credit Event the protection seller pays a Specified Amount (Fixed or (1-Recovery) times the Nominal Principal) to the protection buyer*

See Hull p. 549.

(b) The structure and terms are defined in Hull p. 560ff

(c) This discussion is covered in Hull section 24.9 p. 561. A low level of correlation will make it likely that the losses can be absorbed by the tranches low in the capital structure in most states of the world whereas a high correlation will make it more likely that even high tranches occasionally will be hit. Thus for a given average level of credit risk the equity tranche will benefit from an increase in the perceived correlation and the highest (super senior) tranche will suffer. The effect on mezzanine tranches can be ambiguous.

3. The HJM-model describes the simultaneous evolution of the full term structure of interest rates. Let the evolution of instantaneous forward rates contracted at t for time T be described by the Ito-process

$$df(t,T) = m(t,T,\Omega)dt + s(t,T,\Omega)dz$$

where Ω is a set of state variables.

(a) Under certain conditions we have the following no-arbitrage condition for the drift term:

$$m(t,T,\Omega) = s(t,T,\Omega)\int_t^T s(t,\tau,\Omega)d\tau$$

Comment on this result, and in particular explain under which probability measure it is derived.

(b) As a special case let $s(t,T,\Omega)$ be a constant. Derive the process followed by forward rates. Comment on the distribution of the forward rates.

(c) Certain models of the term structure can be calibrated to be consistent with an intial given term structure. How is this achieved in simple one-factor models as e.g. the Ho-Lee model? How is this achieved in the HJM-model?

Solution:

(a) Cf. Hull, pp. 716-7. What should be commented is that the volatility barring arbitrage determines the drift rate, and that the result presented is derived under the traditional risk neutral probability measure.

(b) By integration we find for this simple model that

$$df(t,T) = s^2(T-t)dt + sdz$$

(this is not derived in the syllabus). This means that the changes to the forward rates, and thus the forward rates, will be normally distributed, which in particular means a positive probability for negative interest rates. It can be noted - but this is not directly in the syllabus (it is in Practice Question 31.3) - that this special case is equivalent to the Ho-Lee-model.

(c) The models that can be made consistent with any (arbitrage free, i.e. having no negative forward rates) initial term structure are termed no-arbitrage models (Hull p. 689ff). This is in models of the short rate achieved by making the drift term dependent on time, e.g. the Ho-Lee model and other named models (Hull p. 690f). In the HJM-model this is directly achieved as the initial value of the full term structure of forward rates (In either case these instantanous rates are not directly observable in the market and must be derived from prices on traded instruments).